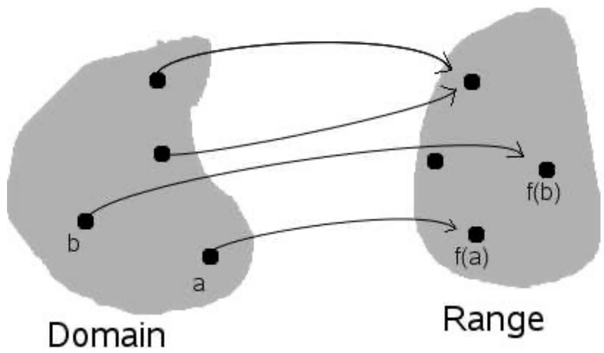
Differential Calculus

Functions

- A **function** is a rule that for each element of one set *D* gives an element of another set *E*. The set *D* is called domain of the function, and *E* is the range.



Example

For each function, graph $y = f(x)$ and determine $f(2)$.

Function	Graph	f(2)
$f(x) = x^2$		

$f(x) = \cos(x)$		
$f(x) = 3^x$		

Derivatives

- The **derivative** of a function $y = f(x)$ describes the rate of change of the function with respect to *x*.
- Graphically, the derivative gives the instantaneous slope of the graph of $f(x)$ at any *x*.
- Notation: $f'(x)$ or $\dfrac{dy}{dx}$ or $\dfrac{d}{dx}f(x)$

- In math, we learned how to determine the slope of a straight line using rise over run, $\dfrac{\Delta y}{\Delta x}$.

How do we determine the slope if we have a curve?

- Recall when analyzing curved position vs. time graphs, we drew a tangent line to determine the slope.

<u>Example</u>

Graph each function and its derivative.

f(x)	**Graph of f(x)**	**Graph of f'(x)**
$f(x) = x$		

$f(x) = x^2$		
$f(x) = x^3$		

Power Rule

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Example

Determine the derivative of each of the following functions.

a) $y = x^2$

b) $y = x^5$

c) $y = \dfrac{1}{x^3}$

d) $y = \sqrt[3]{x}$

Constant Rule

$$\frac{d}{dx}\left(k\right) = 0$$

Constant Multiple Rule

$$\frac{d}{dx}\left(kf(x)\right) = kf'(x)$$

Sum/Difference Rule

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$$

<u>Example</u>

Determine the derivatives of the following polynomials.

a) $y = 2x^5$

b) $y = x^4 + \frac{1}{x^2}$

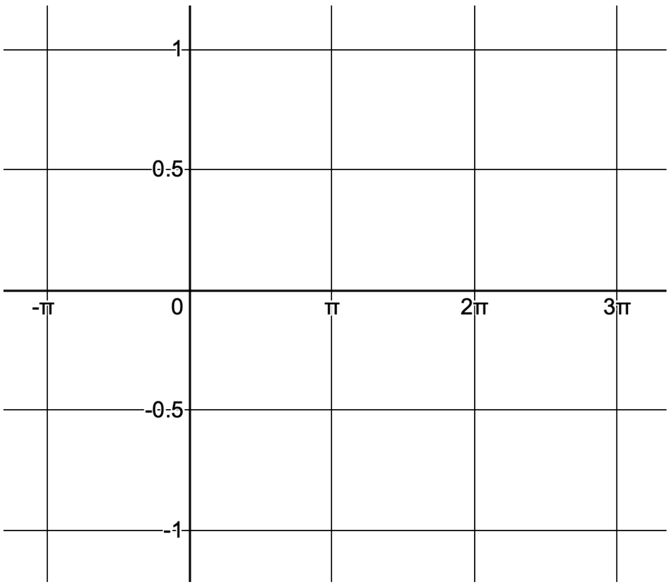
c) $y = 2x^3 - \frac{x^2}{2} + 3x - 5$

Derivatives of Trigonometric Functions

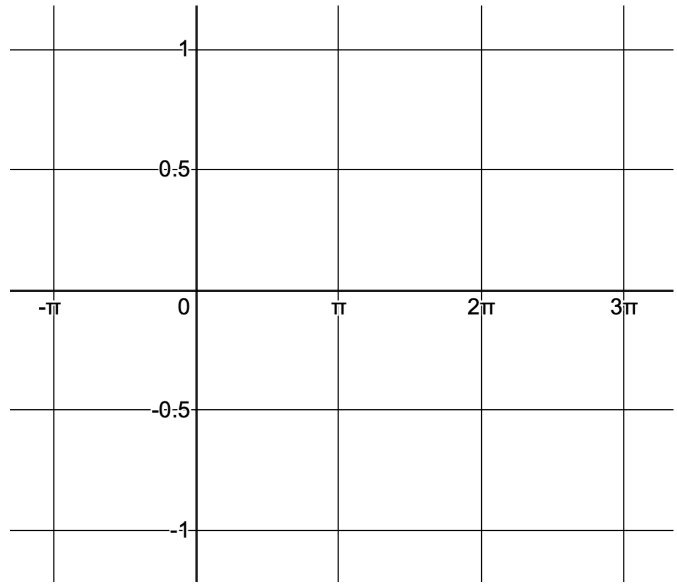
<u>Example</u>

Graph $f(x) = \sin x$ and $g(x) = \cos x$ and their derivatives.

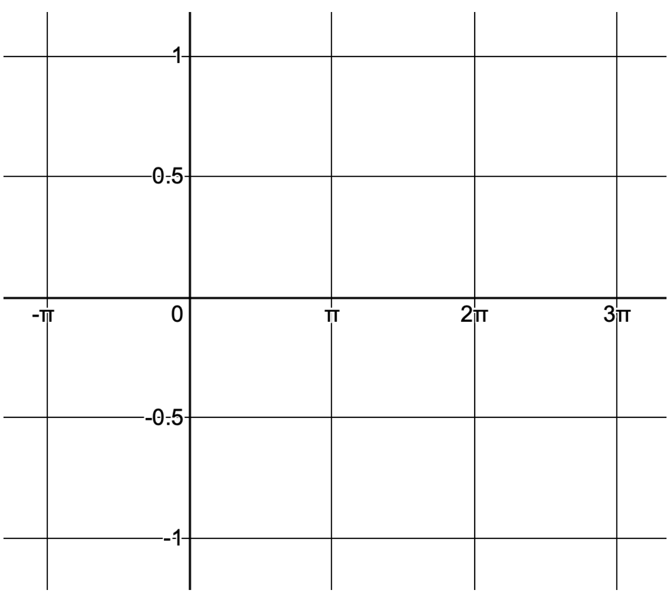
$f(x) = \sin x$



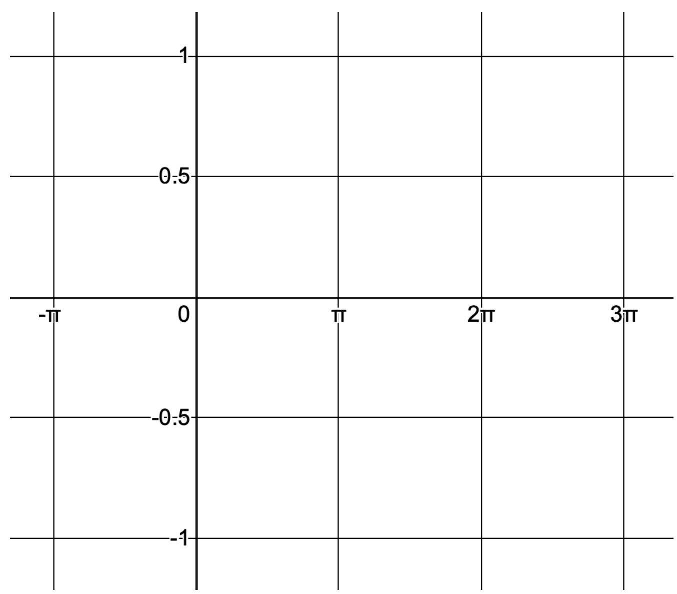
$g(x) = \cos x$



$f'(x)$



$g'(x)$



$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(-\sin x) = -\cos x \qquad \frac{d}{dx}(-\cos x) = \sin x$$

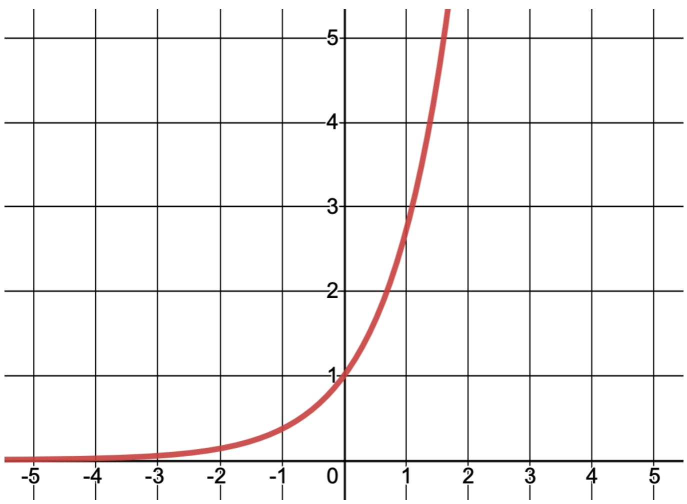
Is there a function (aside from $f(x) = 0$) whose derivative is equal to itself?

Derivatives of Exponential and Logarithmic Functions

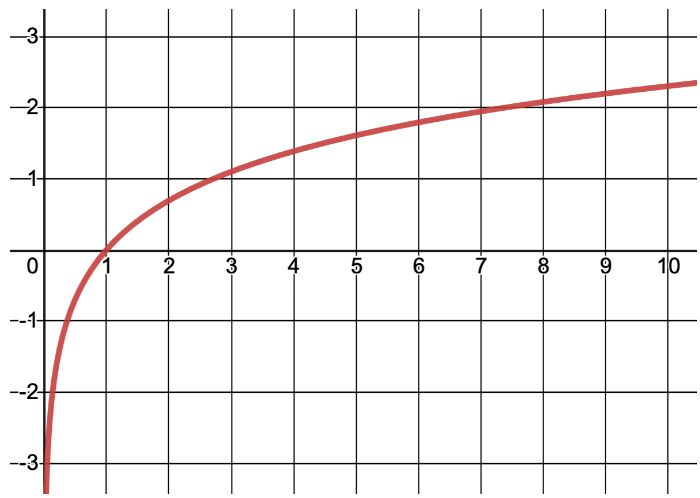
$$\frac{d}{dx}\left(e^x\right) = e^x \qquad\qquad \frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

- Recall that Euler's number *e* is a mathematical constant approximately equal to 2.71828
- Recall that the natural logarithm of a number is its logarithm to base *e* (i.e. $\ln x = \log_e x$).

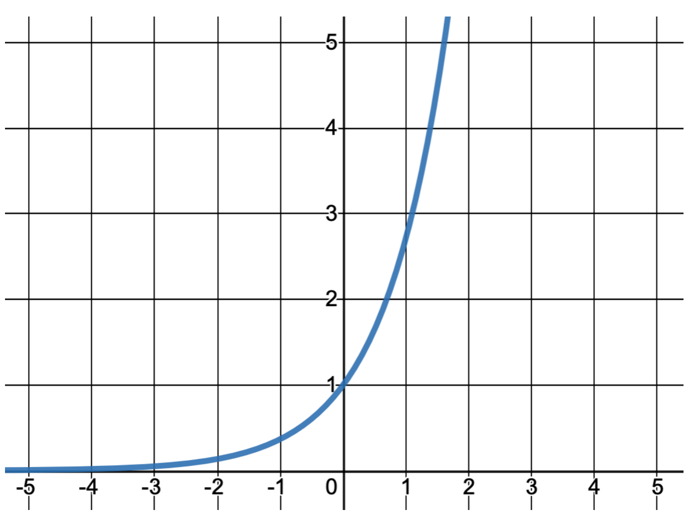
$f(x) = e^x$



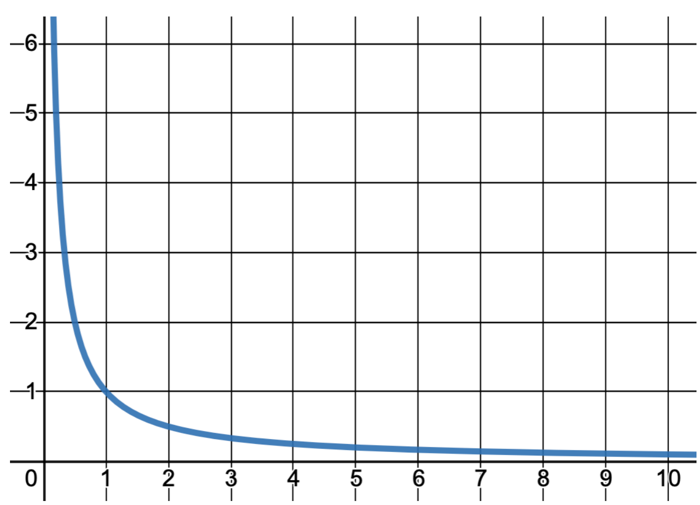
$g(x) = \ln x$



$f'(x) = e^x$



$g'(x) = \frac{1}{x}$



Example

Determine the derivative of each of the following functions.

a) $y = 4e^x$

b) $y = -\frac{\ln x}{5} - 4\cos x$

c) $y = 3e^x - 3x^3 - 3$

d) $y = \frac{4}{\sqrt{x^3}} + 3\sin x + 1$

Product Rule

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)g'(x) + g(x)f'(x)$$

Example

Determine the derivative of the following function using two different methods.

$$y = \left(3x^2 + 5\right)\left(x^3 + 4x\right)$$

Example

Determine the derivative of each of the following functions.

a) $y = 4x^2 \sin x$

b) $y = e^x \cos x$

c) $y = -3 \ln x \left(x^2 - 4x + 1\right)$

d) $y = \dfrac{\cos x}{2x^4} + \sin x - \dfrac{\pi}{2}$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

<u>Example</u>
Determine the derivative of each of the following functions

a) $y = \dfrac{4x + 1}{2x^2 + x}$

b) $y = \dfrac{\ln x}{x}$

c) $y = \csc x$

d) $y = \tan x$

Chain Rule

$$\frac{d}{dx}\left(\left(f(g(x)\right)\right) = f'\left(g(x)\right) \cdot g'(x)$$

<u>Example</u>

Determine the derivative of the following function using three different methods.

$$y = \left(5 - 6x\right)^2$$

<u>Example</u>

Determine the derivative of each of the following functions.

a) $y = \cos\left(2x^3\right)$

b) $y = 5e^{-3t}$

c) $y = 4e^{-3t}\left(\sin 2t + 1\right)$

Example

Prove the quotient rule using the product and chain rules.

Higher Order Derivatives

- The **second derivative** of a function $y = f(x)$ is the derivative of the function's *first* derivative.
- Graphically, the derivative gives the concavity of the graph of $f(x)$ at any *x*.
- Notation: $f''(x)$ or $\dfrac{d^2y}{dx^2}$ or $\dfrac{d^2}{dx^2}f(x)$
- $f'''(x) = \left(f''(x)\right)'$ is the *third* derivative and $f^{(4)}(x) = \left(f'''(x)\right)'$ is the *fourth* derivative.

<u>Example</u>

Determine the second derivative of each of the following functions.

a) $y = 6x^4 - 2x^3 + 10x^2 - 4$

b) $y = 4\sin(3x)$

c) $y = 2e^x - 4\sqrt{x}$

d) $y = \left(3x^2 + e^x\right)^2$

Applications of Derivatives in Physics

- Derivatives are used whenever you are looking for the rate of change of a quantity (or slope of a graph).
  - Velocity is the rate of change of position

$$v = \frac{dx}{dt}$$

  - Acceleration is the rate of change of velocity

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

  - Force is the rate of change of momentum

$$F = \frac{dp}{dt}$$

  - Power is the rate at which work is done

$$P = \frac{dE}{dt}$$