Integral Calculus

What is $2x^3$ *the derivative of?*

Antiderivatives and Indefinite Integrals

- Given a function $f(x)$, an **antiderivative** of $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$.
- If $F(x)$ is any antiderivative of $f(x)$, then the most general antiderivative of $f(x)$ is called the **indefinite integral** and denoted

$$\int f(x)dx = F(x) + C$$

integral symbol

integrand

constant of integration

variable of integration

Constant Multiple Rule

$$\int kf(x)dx = k\int f(x)dx$$

Sum/Difference Rule

$$\int \left(f(x) \pm g(x)\right) dx = \int f(x)dx \pm \int g(x)dx$$

Example

Evaluate the following indefinite integrals

a) $\int x^5 dx$

b) $\int \frac{2}{x^2} dx$

c) $\int x^n dx$

d) $\int \sin x dx$

e) $\int \cos x dx$

f) $\int e^{ax} dx$

g) $\int \frac{1}{x} dx$

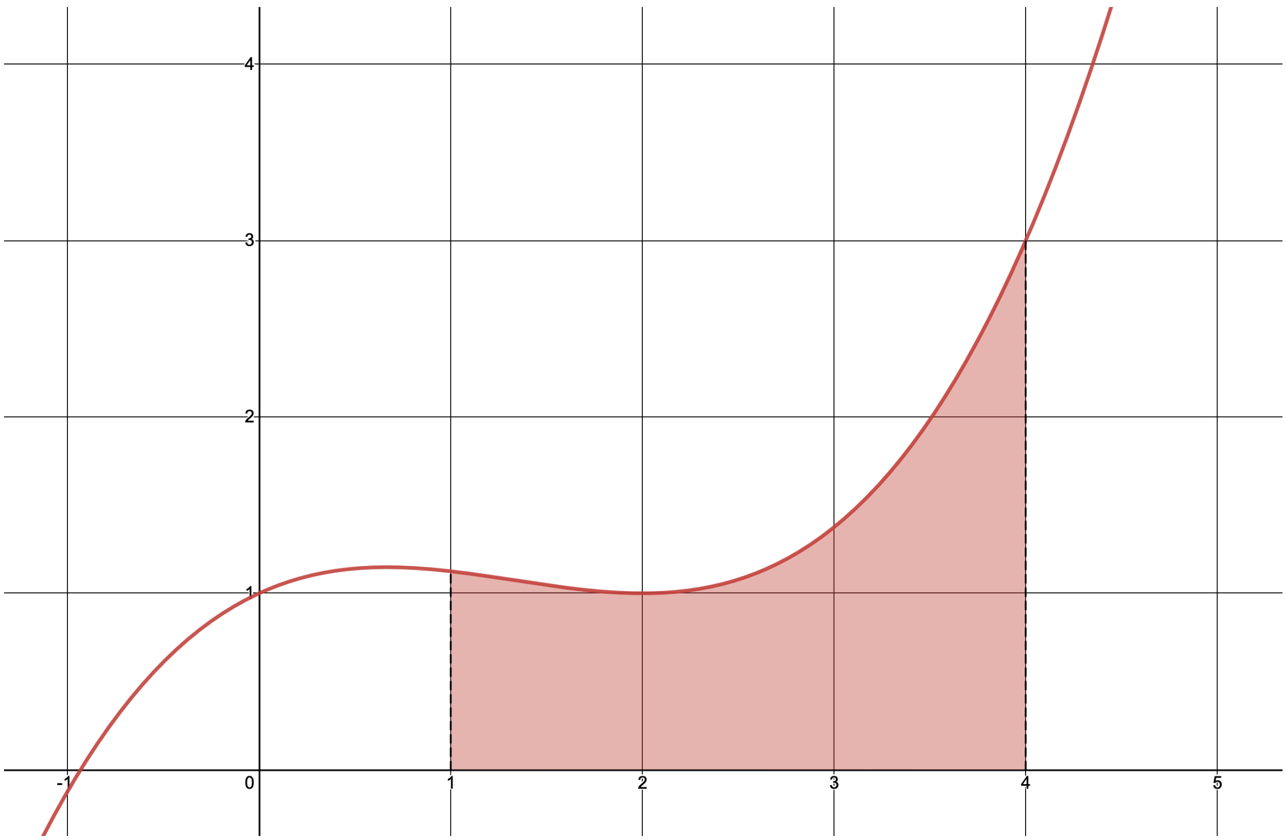
h) $\int \left(e^{-x} - \sin(2x)\right) dx$

- The constant of integration can be determined with the initial (or other given) conditions.

<u>Example</u>

If $f'(x) = 6x^2 - 1$ and $f(2) = 10$, what is $f(x)$?

How would you determine the area of the shaded region below?



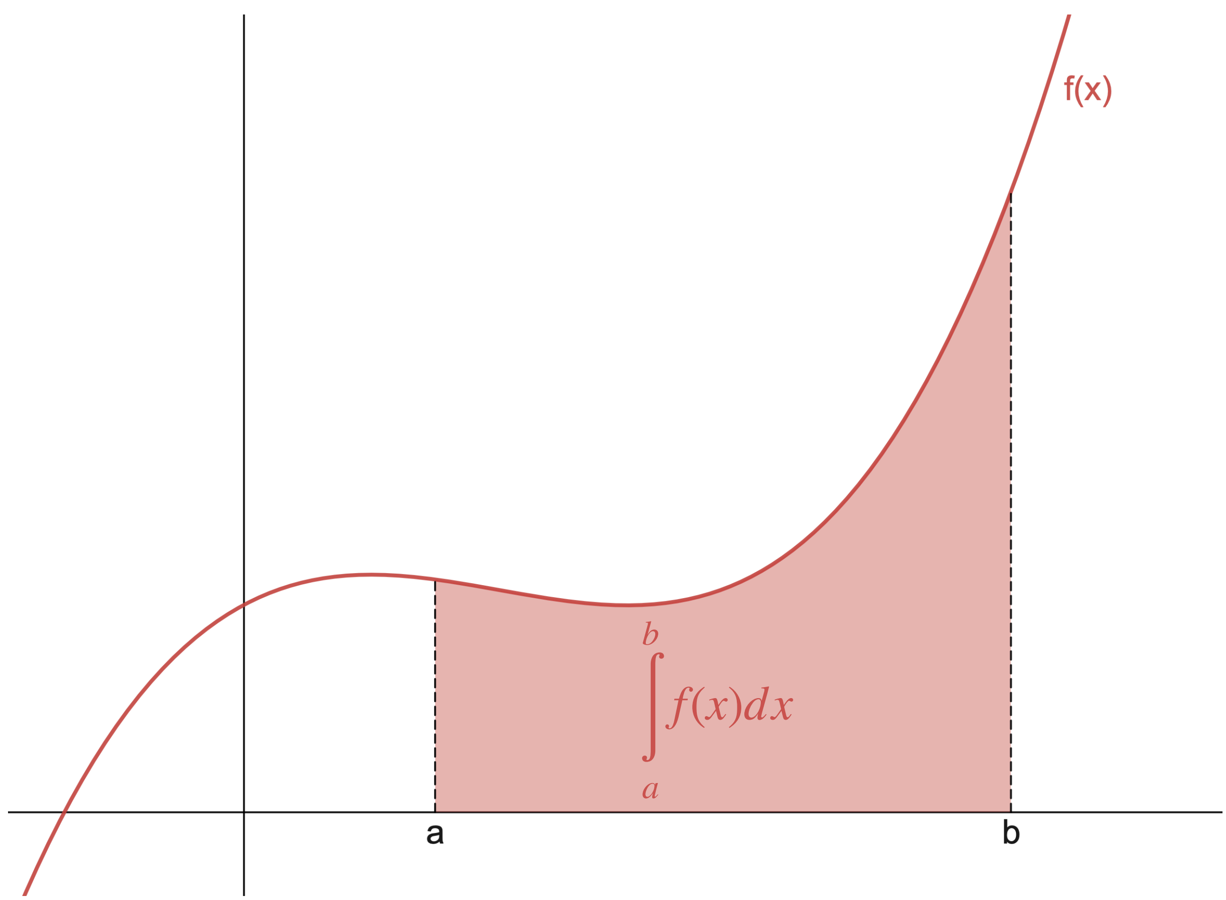
Definite Integrals

- Suppose $f(x)$ is a continuous function on [*a*, *b*] and also suppose that $F(x)$ is any antiderivative for $f(x)$. Then the **definite integral** of $f(x)$ is

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

limits of integration

- Graphically, the definite integral gives the area under the curve $y = f(x)$ (i.e. between $f(x)$ and the *x*-axis) on the interval [*a*, *b*].



Properties of the Definite Integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

<u>Example</u>

Find the area under $f(x) = x^2$ between $x = 0$ and $x = 2$.

Example

Determine the area under $f(\theta) = \sin\theta$

a) between $\theta = 0$ and $\theta = \pi$

b) between $\theta = 0$ and $\theta = 2\pi$

Integration by Substitution

- **Integration by substitution** (also known as *u*-substitution) is the reverse of the chain rule used in differentiation.
- Consider an integral which can be written in the following form

$$\int f\left(\underbrace{g(x)}_{u}\right)\underbrace{g'(x)dx}_{du}$$

- Notice how you have an inner function $g(x)$ and its derivative $g'(x)$.
- Let $u = g(x)$. Then $\frac{du}{dx} = g'(x) \rightarrow du = g'(x)dx$. After substituting, our integral is

$$\int f(u)du$$

- After integrating, return the function to be in terms of the original variable.

Example

Evaluate the following integrals.

a) $\int (8x - 12)\left(4x^2 - 12x\right)^4 dx$

b) $\int 90x^2 \sin\left(2 + 6x^3\right) dx$

c) $\int_0^1 (3 - 4x)\left(4x^2 - 6x + 1\right)^{10} dx$

Applications of Integrals in Physics

- Definite integrals can be used to determine the sum of many infinitesimal parts (the area under a curve).
  - Displacement is the area under a velocity vs. time graph.

$$d = \Delta x = \int_{t_1}^{t_2} v(t)dt$$

  - *Change in* velocity is area under an acceleration vs. time graph.

$$\Delta v = \int_{t_1}^{t_2} a(t)dt$$

  - Work is the area under a force vs. position graph.

$$W = \Delta E = \int_{x_1}^{x_2} F(x)dx$$

  - Impulse is the area under a force vs. time graph.

$$J = \Delta p = \int_{t_1}^{t_2} F(t)dt$$