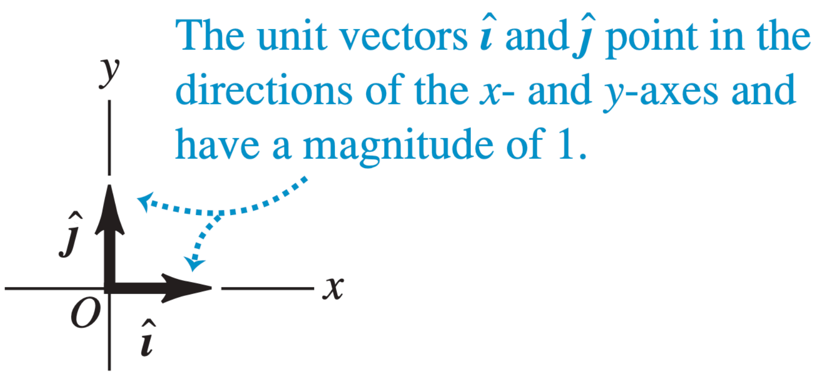
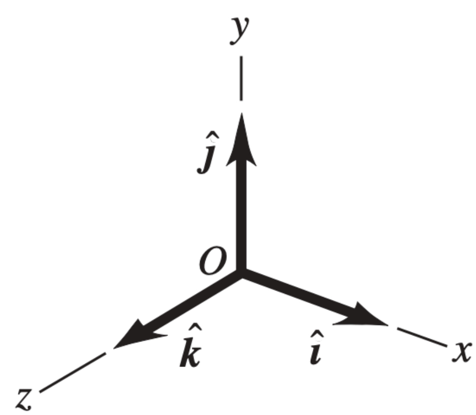
Vector Arithmetic

Unit Vectors

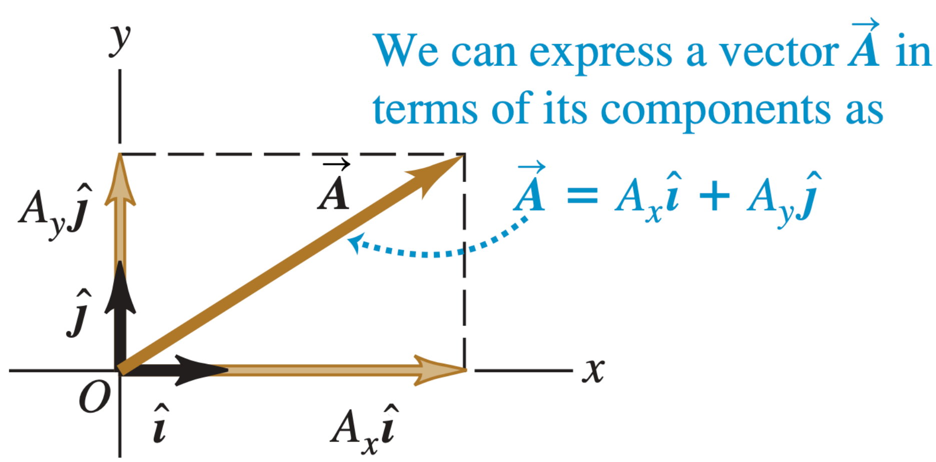
- A **unit vector** is a vector that has a magnitude of 1 and is used to describe a direction.
- The standard unit vectors in the direction of the *x*, *y* and *z* axes of a three-dimensional Cartesian coordinate system are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.





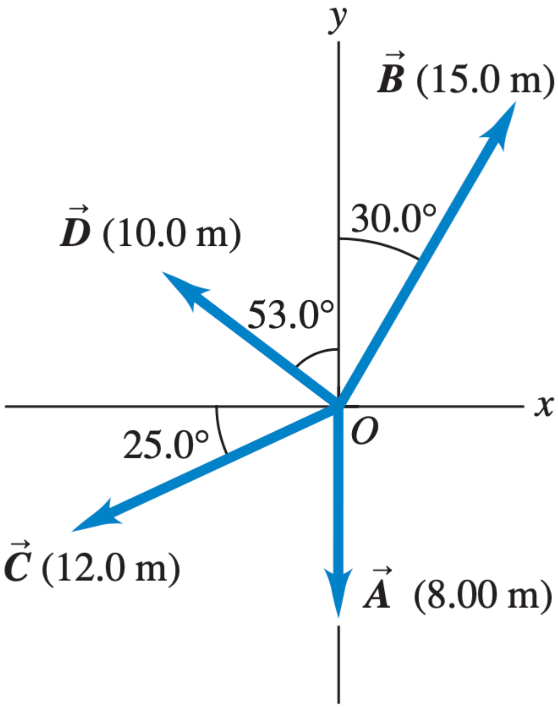
- We can write a vector $\overrightarrow{A}$ in terms of its components as

$$\overrightarrow{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$



Example

Write each vector in the figure below in terms of the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$.



Vector Addition

- Using unit vectors, we can express the vector sum $\overrightarrow{R}$ of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ as follows:

$$\begin{aligned}\overrightarrow{R} &= \overrightarrow{A} + \overrightarrow{B} \\ &= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right) \\ &= \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} + \left(A_z + B_z\right)\hat{\mathbf{k}}\end{aligned}$$

Example

Given two vectors

$\overrightarrow{A} = 3\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\overrightarrow{B} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$,

a) determine $\overrightarrow{A} + \overrightarrow{B}$

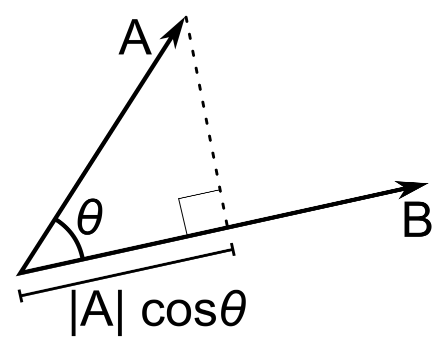
b) determine $\overrightarrow{B} - \overrightarrow{A}$

Dot (Scalar) Product

- The **dot product** of two vectors $\vec{A}$ and $\vec{B}$ is defined by

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right| \left|\vec{B}\right| \cos\theta$$

  where $\left|\vec{A}\right|$ denotes the magnitude of $\vec{A}$ and θ is the angle between $\vec{A}$ and $\vec{B}$.
- The dot product is thus the magnitude of one vector multiplied by the component of a second vector in the direction of the first.
- Note that the dot product of two perpendicular vectors is zero.



$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$$

- Using unit vectors, we can express the dot product of two vectors $\vec{A}$ and $\vec{B}$ as follows:

$$\begin{aligned}\vec{A} \cdot \vec{B} &= \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) \cdot \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right) \\ &= A_x\hat{\mathbf{i}} \cdot B_x\hat{\mathbf{i}} + A_x\hat{\mathbf{i}} \cdot B_y\hat{\mathbf{j}} + A_x\hat{\mathbf{i}} \cdot B_z\hat{\mathbf{k}} \\ &\quad + A_y\hat{\mathbf{j}} \cdot B_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} \cdot B_y\hat{\mathbf{j}} + A_y\hat{\mathbf{j}} \cdot B_z\hat{\mathbf{k}} \\ &\quad + A_z\hat{\mathbf{k}} \cdot B_x\hat{\mathbf{i}} + A_z\hat{\mathbf{k}} \cdot B_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} \cdot B_z\hat{\mathbf{k}} \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

- Applications of the dot product:
  - $W = \overrightarrow{F} \cdot \overrightarrow{d}$
  - $\Phi_B = \overrightarrow{B} \cdot \overrightarrow{A}$

<u>Example</u>

Given two vectors

$\overrightarrow{A} = 3\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\overrightarrow{B} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$,

a) determine $\overrightarrow{A} \cdot \overrightarrow{B}$

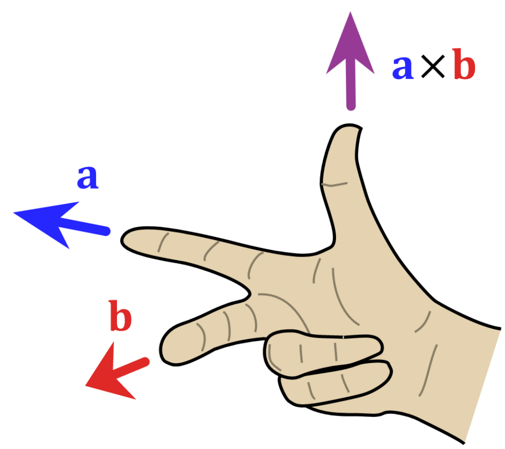
b) determine $\overrightarrow{B} \cdot \overrightarrow{A}$

Cross (Vector) Product

- The **cross product** of two vectors $\vec{A}$ and $\vec{B}$ is defined by a vector with a direction perpendicular to both $\vec{A}$ and $\vec{B}$ and a magnitude equal to

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

where $\left|\vec{A}\right|$ denotes the magnitude of $\vec{A}$ and θ is the angle between $\vec{A}$ and $\vec{B}$.
- The magnitude of the cross product is thus the magnitude of one vector multiplied by the perpendicular component of a second vector.
- Note that the cross product of two parallel vectors is zero.
- The direction of the cross product is given by the right-hand rule.



- Applying the definition of the cross product to the unit vectors, we get

$$\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \mathbf{0} \qquad \hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}$$
$$\hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}} \qquad \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \mathbf{0} \qquad \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}} \qquad \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \mathbf{0}$$

- Using unit vectors, we can express the cross product of two vectors $\vec{A}$ and $\vec{B}$ as follows:

$$\vec{A} \times \vec{B} = \left(A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}\right) \times \left(B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}\right)$$
$$= A_x \hat{\mathbf{i}} \times B_x \hat{\mathbf{i}} + A_x \hat{\mathbf{i}} \times B_y \hat{\mathbf{j}} + A_x \hat{\mathbf{i}} \times B_z \hat{\mathbf{k}}$$
$$+ A_y \hat{\mathbf{j}} \times B_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} \times B_y \hat{\mathbf{j}} + A_y \hat{\mathbf{j}} \times B_z \hat{\mathbf{k}}$$
$$+ A_z \hat{\mathbf{k}} \times B_x \hat{\mathbf{i}} + A_z \hat{\mathbf{k}} \times B_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}} \times B_z \hat{\mathbf{k}}$$

$$\vec{A} \times \vec{B} = \left(A_y B_z - A_z B_y\right) \hat{\mathbf{i}} + \left(A_z B_x - A_x B_z\right) \hat{\mathbf{j}} + \left(A_x B_y - A_y B_x\right) \hat{\mathbf{k}}$$

- The cross product can be expressed in determinant form as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- Applications of the cross product:
  - $\vec{\tau} = \vec{r} \times \vec{F}$
  - $\vec{F}_B = q\vec{v} \times \vec{B}$
  - $\vec{L} = \vec{r} \times \vec{p}$

Example

Given two vectors

$\overrightarrow{A} = 3\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\overrightarrow{B} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$,

a) determine $\overrightarrow{A} \times \overrightarrow{B}$

b) determine $\overrightarrow{B} \times \overrightarrow{A}$

Example

Given two vectors $\overrightarrow{A} = 6\hat{\mathbf{i}} + 8\hat{\mathbf{j}}$ and $\overrightarrow{B} = -5\hat{\mathbf{i}} + 12\hat{\mathbf{j}}$,

a) determine the magnitude and direction of $\overrightarrow{A}$ and $\overrightarrow{B}$

b) determine $\overrightarrow{A} + \overrightarrow{B}$

c) determine $\overrightarrow{B} - \overrightarrow{A}$

d) use two methods to determine $\overrightarrow{A} \cdot \overrightarrow{B}$

e) use two methods to determine $\overrightarrow{A} \times \overrightarrow{B}$